Optical soliton in the presence of perturbations

1993 J. Phys. A: Math. Gen. 26 3869

(http://iopscience.iop.org/0305-4470/26/15/035)

J. Phys. A: Math. Gen. 26 (1993) 3869–3882. Printed in the UK

Optical soliton in the presence of perturbations

A Rybin and J Timonen

Department of Physics, University of Jyväskylä, PO BOX 35, SF-40351 Jyväskylä, Finland

Received 28 September 1992, in final form 12 May 1993

Abstract. In the framework of the reduced Maxwell–Bloch (RMB) system we investigate in local form the interaction of optical soliton with an arbitrary background field composed of causal and spontaneous parts. A non-trivial phase shift of the soliton is also found.

1. Introduction

A long standing problem in nonlinear optics has been that of properly understanding the behaviour of an ultrashort pulse of light in a dispersive medium (see, e.g., [1]). Despite the significant progress [1, 2, 3] made in this problem during the last two decades there are still interesting questions concerning interaction of localised excitations (solitons) with irregular background disturbances (noise) or following the terminology of the theory of scattering, the interaction of soliton with continuum spectrum. Such an interaction is of general physical significance. In nonlinear optics in particular these processes are of great importance in the theory of optical systems for information transfer for example, where noise can interfere with localized information bearing signals. From a formal point of view such noise is a solution of the considered soliton bearing nonlinear equation. Thus the problem of interaction is that of nonlinear superposition of soliton and noise. The approach we report in this work is applicable to any nonlinear system as far as the system is completely integrable. This in a pragmatical sense means that the considered nonlinear system is the compatibility condition of an associated linear system (zero curvature condition). Technically our approach consists of constructing a solution of this linear system corresponding to arbitrary background fields. This solution is found in the form of an asymptotic series with respect to the spectral parameter. Given this solution we can then apply a dressing procedure (in our work the Darboux transformation method) to construct a nonlinear superposition of a soliton and the background in the local form, i.e. at a finite spacetime point. We may think of our approach as a variant of perturbation theory in the sense that perturbation of the exact solution is not accounted for by an extra term in the nonlinear equation but by the arbitrary background. A small parameter of this perturbation theory is the reciprocal of the spectral parameter. In order to exemplify the general method we have chosen to consider a system describing an interaction of an electric field with an active medium composed of two-level atoms.

We assume that the dispersive medium consists of two-level atoms with resonance frequency ω_0. Usually [2, 3] the interaction of light with a two-level medium has been described in terms of the Maxwell–Bloch (MB) equations

$$\varepsilon_\xi = \langle\rho\rangle \qquad \rho\tau + 2\mathrm{i}\eta\rho = \mathcal{N}\varepsilon \qquad \mathcal{N}_t = -\tfrac{1}{2}(\bar{\varepsilon}\rho + \varepsilon\bar{\rho}) \tag{1.1}$$

for the complex, slowly varying amplitudes ε, ρ and $\mathcal{N}$ of the electric field, polarization and inversion, respectively. Here $\tau = \Omega_1(t - x/c)$, $\xi = \Omega_1 x/c$, $\Omega_1 = 2\pi n_0\omega_0\mu^2/\hbar$, $\eta = (\omega - \omega_0)/2\Omega_1$, n_0 is the density of atoms in the active medium, μ is the dipole momentum matrix element of the two-level atoms, $\bar{\mathcal{O}}$ is the complex conjugate of $\mathcal{O}$ and $\langle\cdot\rangle = \int_{-\infty}^{\infty} g_1(\eta)(\cdot)\,\mathrm{d}\eta$ an average over inhomogeneous broadening. Subscripts denote partial derivatives.

It would be better, however, to work directly with dynamical variables including their rapidly oscillating parts, and therefore we shall adopt here the method described in [1]. This method which leads to the so-called reduced Maxwell–Bloch (RMB) system, has also other advantages over the usual MB system (see [1]). The RMB system is described by equations

$$u_t = -\omega v \qquad v_t = \omega u + En \qquad n_t = -Ev \qquad E_z + E_t = \alpha\langle\omega v\rangle \tag{1.2}$$

where E is a real electric field, ω is the resonance frequency, $n(\omega, t, z)$ the density of excited atoms in the dispersive (active) medium (i.e. inversion) and $u(\omega, t, z)$, $v(\omega, t, z)$ are the two components of the polarization. The symbol $\langle\cdot\rangle = \int_0^{\infty} g_2(\omega)(\cdot)\,\mathrm{d}\omega$ denotes an average over inhomogeneous broadening. Also, $z = x/c$ and $\alpha = 4\pi n_0\mu^2/\hbar$.

The problem to be solved is defined by specifying the initial and boundary conditions for the set of equations (1.2). A suitably general formulation of a similar problem was given in [3].

At $t = -\infty$ the active medium is assumed to be in a state with a given inversion $n(\omega, t = -\infty, z) = n_-(\omega, z)$ and polarization $\sigma(\omega, t = -\infty, z) = \sigma_-(\omega, z)$. An ultrashort optical pulse is then introduced such that its time dependence at $z = 0$ is specified, $E(t, z = 0) = E_0(t)$. The radiation field in the medium is therefore a nonlinear superposition of two parts, the 'spontaneous' part induced by the initial polarization at $t = -\infty$, and the 'causal' part induced by the ultrashort optical pulse. The interaction we shall study in detail is that between an optical soliton and an arbitrary background field which consists of the two parts described above.

In section 2 we shall first re-examine the results of [3] using the formalism of the matrix Riemann–Hilbert problem in distinction from [3] where the Gelfand-Levitan formalism was used. The advantage of the present approach is that we can restore all components of the solution including those which are needed to describe the quantum system. Let us comment on this point in more detail. In the case when either the RMB system or the MB system has a spectral line of final width, the components of the density matrix related to the solution for the electric field cannot be restored from the nonlinear system itself due to inhomogeneous broadening. In this respect the MB system is even worse than the RMB system because exact resonance corresponds to the choice $\eta = 0$ and $g_1(\eta) = \delta(\eta)$ in the MB system, and $\omega = \omega_0$ and $g_2(\omega) = \delta(\omega - \omega_0)$ in the RMB system. Thus for the MB system it is more difficult to trace the points where frequency and inhomogeneous broadening factors should be inserted while deducing the correct form of the solution from a particular solution found for the sharp line case. This observation provides the motivation for deriving formulae for all components of the solution. Furthemore, it is interesting to see how the matrix Riemann–Hilbert formalism works in the case of a nonlinear system with a singular dispersion relation which is the case for the RMB system. Of course, up to minor modifications our analysis is completely applicable to the MB system as well. The second objective of section 2 is to clarify the analytical structure of the solution of the auxiliary linear system in the case of arbitrary background fields (coefficients of the linear problem). This solution will then be used section 4 to construct the one-soliton solution on an arbitrary background.

2. General solution of the reduced Maxwell–Bloch system

The set of equations (1.2) is the compatibility condition for the (auxiliary) linear problem

$$\Phi_t = U\Phi \tag{2.1a}$$

$$\Phi_z = V\Phi \tag{2.1b}$$

with

$$U = -\frac{\mathrm{i}\lambda}{2}\sigma_3 + \frac{\mathrm{i}}{2}E\sigma_1 \tag{2.2a}$$

$$V = -U - \mathrm{i}\frac{\alpha}{4}\left\langle\!\!\left\langle \omega\frac{\hat{\rho}}{\omega-\lambda} \right\rangle\!\!\right\rangle \tag{2.2b}$$

where σ_j, $j = 1, 2, 3$ are the usual Pauli spin matrices, λ is the spectral parameter and

$$\hat{\rho} = \begin{pmatrix} n & Q \\ \bar{Q} & -n \end{pmatrix} \qquad Q = u - \mathrm{i}w\,. \tag{2.3}$$

By double angular brackets we denote average over symmetrised broadening:

$$\langle\!\langle \cdot \rangle\!\rangle = \int_{-\infty}^{\infty} g^s(\omega)(\cdot)\,\mathrm{d}\omega \qquad g^s(\omega) = g_2(\omega) + g_2(-\omega).$$

We shall assume in what follows that initial pulses (potentials of the Zakharov–Shabat eigenvalue problem) are rapidly decreasing functions of time

$$\int_{-\infty}^{\infty} |E_0(t)|\,\mathrm{d}t < \infty.$$

First we shall find the z-dependence of the monodromy matrix T which we define in the usual way as

$$T(\lambda, z) = \begin{pmatrix} a(\lambda, z) & -\bar{b}(\lambda, z) \\ b(\lambda, z) & \bar{a}(\lambda, z) \end{pmatrix} \tag{2.4}$$

so that it satisfies the symmetry properties of the auxiliary problem. Jost functions $T_\pm$ are connected through

$$T_-(\lambda, t) = T_+(\lambda, t)T(\lambda) \qquad \operatorname{Im}\lambda = 0\,. \tag{2.5}$$

Note that $T_\pm$ can be expressed in the form $\left(T_\pm^{(1)}, T_\pm^{(2)}\right)$, where $T_\pm^{(1,2)}$ are two-component vectors. The matrix $U(\lambda, t, z)$ is found to satisfy two involution conditions, namely

$$\sigma_1 U(\lambda)\sigma_1 = U(-\lambda) \qquad \bar{U}(\lambda) = \sigma_2 U(\bar{\lambda})\sigma_2 \tag{2.6}$$

which means that the elements of the monodromy matrix have the following properties:

$$\bar{b}(\lambda) = -b(-\lambda) \qquad \operatorname{Im}\lambda = 0 \tag{2.7a}$$

$$a(\lambda) = \bar{a}(-\bar{\lambda})\,. \tag{2.7b}$$

It follows immediately from the symmetry relations (2.7*b*) that the zeros λ_j of $a(\lambda)$, i.e. bound states of the auxiliary scattering problem, arise on the imaginary λ axis and as the pairs $(\lambda_k, -\bar{\lambda}_k)$ in the complex λ plane

$$\begin{aligned} \lambda_j &= \mathrm{i}\kappa_j && j = 1, 2, \ldots, n_1 \\ \lambda_{k+n_2} &= -\bar{\lambda}_k && k = n_1 + 1, \ldots, n_1 + n_2 . \end{aligned} \tag{2.8}$$

Here κ_j, $\mathrm{Im}\,\lambda_k$ and $\mathrm{Re}\,\lambda_k > 0$. Total number of zeros n is equal to $n = n_1 + 2n_2$.

Through the zeros of $a(\lambda)$ we can define the so-called transition coefficients for the discrete spectrum γ_j such that

$$T_-^{(1)}(z, \lambda_j) = \gamma_j T_+^{(2)}(z, \lambda_j) \qquad j = 1, \ldots, n . \tag{2.9}$$

They are readily found to satisfy

$$\begin{aligned} \gamma_j &= -\bar{\gamma}_j && j = 1, 2, \ldots, n_1 \\ \gamma_k &= -\bar{\gamma}_{k+n_2} && k = n_1, \ldots, n_1 + n_2 . \end{aligned} \tag{2.10}$$

We can derive the 'evolution equations' of the scattering data by looking at the asymptotic behaviour in time of the relevant variables. As $t \to \pm\infty$

$$\hat{\rho}(\omega, t, z) \to E(\omega, t)\hat{\rho}^{\pm}(\omega, z)E^{-1}(\omega, t)$$

with

$$\begin{aligned} \hat{\rho}^{\pm}(\omega, z) &= \begin{pmatrix} n_{\pm}(\omega, z) & \sigma_{\pm}(\omega, z) \\ \bar{\sigma}_{\pm}(\omega, z) & -n_{\pm}(\omega, z) \end{pmatrix} \\ E(\omega, t) &= \exp\left(-\tfrac{1}{2}\mathrm{i}\omega t \sigma_3\right) . \end{aligned} \tag{2.11}$$

From physical arguments it is evident that matrices $\hat{\rho}^{\pm}(\omega, z)$ are not independent. By applying the monodromy matrix we find that

$$\begin{aligned} n_+ &= n_-(|a|^2 - |b|^2) - \bar{a}\bar{b}\bar{\sigma}_- - ab\sigma_- \\ \sigma_+ &= 2a\bar{b}n_- + a^2\sigma_- - \bar{b}^2\bar{\sigma}_- . \end{aligned} \tag{2.12}$$

This result means that the 'evolution equation' of the monodromy matrix $T(\lambda, z)$ is

$$\frac{\mathrm{d}}{\mathrm{d}z}T(\lambda, z) = \tilde{V}_+(\lambda, z)T(\lambda, z) - T(\lambda, z)\tilde{V}_-(\lambda, z) \tag{2.13}$$

where

$$\tilde{V}_{\pm} = \lim_{t \to \pm\infty} E^{-1}(\lambda, t)V(\lambda, t, z)E(\lambda, t) \qquad \mathrm{Im}\,\lambda = 0 . \tag{2.14}$$

An interesting observation can readily be made from (2.13): these equations lead to an explicit solution for the reflection coefficient $r(\lambda, z)$

$$r(\lambda, z) = \frac{\bar{b}(\lambda, z)}{a(\lambda, z)} \tag{2.15}$$

in terms of the 'causal' contribution $r_c(\lambda, z)$ and the 'spontaneous' contribution $r_s(\lambda, z)$ such that

$$\begin{aligned} r &= r_c + r_s \\ r_c(\lambda, z) &= r(\lambda, 0)\exp(i\chi(\lambda, z)) \\ r_s(\lambda, z) &= \frac{\alpha\pi\lambda}{2} g^s(\lambda) \int_0^z \sigma_-(\lambda, z') \exp\big(i\chi(\lambda, z) - i\chi(\lambda, z')\big)\, dz' . \end{aligned} \tag{2.16a}$$

Here

$$\chi(\lambda, z) = \lambda z - \frac{\alpha}{2} \int_0^z \left\langle\!\!\left\langle \omega \frac{n_-}{\omega - \lambda - i0} \right\rangle\!\!\right\rangle dz' . \tag{2.16b}$$

With the help of (2.15) and (2.16), equations (2.13) admit solutions for the scattering data in the form

$$\begin{aligned} b(\lambda, z) = \bigg(b(\lambda, 0) + \frac{\alpha\pi\lambda}{2} g^s(\lambda)\bar{a}(\lambda, 0) \int_0^z \bar{\sigma}_-(\lambda, z') \exp(i\bar{\chi}(\lambda, z'))\, dz' \bigg) & \\ \times \exp(-i\bar{\chi}(\lambda, z) + \Omega_-(\lambda, z)) & \end{aligned} \tag{2.17a}$$

$$a(\lambda, z) = a(\lambda, 0)\exp(-\Omega_+(\lambda, z)) \tag{2.17b}$$

$$\gamma_j(z) = \gamma_j(0)\exp\big(-i\chi(\lambda_j, z) + \Omega(\lambda_j, z)\big) \tag{2.17c}$$

where

$$\Omega(\lambda, z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\omega \frac{1}{\omega - \lambda} \ln\left(\frac{1 + |r(\omega, z)|^2}{1 + |r(\omega, 0)|^2}\right) \tag{2.18}$$

and $\Omega_+(\Omega_-)$ is the boundary value of Ω on the real λ axis from above (below). The solution of the Cauchy problem for the RMB system can now be reduced to solution of the matrix Riemann–Hilbert problem,

$$G(\lambda, t, z) = G_+(\lambda, t, z)G_-(\lambda, t, z) \tag{2.19a}$$

for the G-matrix

$$G(\lambda, t, z) = \begin{pmatrix} 1 & -\bar{b}e^{-i\lambda t} \\ -be^{i\lambda t} & 1 \end{pmatrix} = E(\lambda, t)G_0(\lambda, z)E^{-1}(\lambda, t) \tag{2.19b}$$

with $\operatorname{Im}\lambda = 0$. Here $G_\pm$ are [4]

$$\begin{aligned} G_-(\lambda, t, z) &= S_-(\lambda, t, z)E^{-1}(\lambda, t) \\ G_+(\lambda, t, z) &= a(\lambda, z)E(\lambda, t)S_+^{-1}(\lambda, t, z) \end{aligned} \tag{2.19c}$$

and

$$\begin{aligned} S_- &= \big(T_+^{(1)}, T_-^{(2)}\big) \\ S_+ &= \big(T_-^{(1)}, T_+^{(2)}\big) . \end{aligned} \tag{2.19d}$$

In order to establish in detail the connection between the solution $G_\pm$ of the Riemann–Hilbert problem and that of the RMB system, i.e. $E(t,z)$ and $\hat{\rho}(\omega,t,z)$ which follow from the initial pulse $E_0(t)$ and from the state of the medium at $t=-\infty$ as described by $n_-(\omega,z)$ and $\sigma_-(\omega,z)$, we need first to derive a number of preliminary results.

To begin with we first reformulate the factorization problem (2.19) such that the matrix G is expressed in the form

$$G(\lambda,t,z)=E(\lambda,t)\mathrm{e}^{-\Psi_+(\lambda,z)}\tilde{G}(\lambda,z)\mathrm{e}^{\Psi_-(\lambda,z)}E^{-1}(\lambda,t) \tag{2.20}$$

and the matrices $G_\pm$ are replaced by $F_\pm(\lambda,t,z)$ which are defined as

$$\begin{aligned} F_+ &= G_+^{-1}E(\lambda,t)\exp\left(-\Psi_+(\lambda,z)\right) \\ F_- &= G_-E(\lambda,t)\exp\left(-\Psi_-(\lambda,z)\right) . \end{aligned} \tag{2.21}$$

Here $\Psi(\lambda,z)$ is a piecewise analytical matrix function which is to be determined, and $\Psi_\pm$ are its boundary values at the real λ axis from above (+) and below (−).

In terms of $F_\pm(\lambda,t,z)$ the conjugation problem (2.19*a*) takes the form

$$F_-=F_+\tilde{G} . \tag{2.22}$$

By differentiating $F_+^{-1}F_-=\tilde{G}$ with respect to z we find that

$$\frac{\mathrm{d}}{\mathrm{d}z}\tilde{G}=-F_+^{-1}(V_+-V_-)F_- \tag{2.23}$$

where $V_\pm$ are again the values in the limit $\mathrm{Im}\,\lambda\to 0^\pm$ of matrix V defined in (2.2*b*). In the limit $t\to-\infty$ and for $\mathrm{Im}\,\lambda=0$, $G_\pm$ can be shown to have the asymptotic forms [4]

$$\begin{aligned} G_+^{-1} &= \begin{pmatrix} 1/a(\lambda) & [\bar{b}(\lambda)/a(\lambda)]\mathrm{e}^{-\mathrm{i}\lambda t} \\ 0 & 1 \end{pmatrix} + \mathrm{o}(1) \\ G_- &= \begin{pmatrix} \bar{a}(\lambda) & 0 \\ -b(\lambda)\mathrm{e}^{\mathrm{i}\lambda t} & 1 \end{pmatrix} + \mathrm{o}(1) . \end{aligned} \tag{2.24}$$

Combining this result with the asymptotic behaviour at $t\to-\infty$ of (2.23), we find that

$$\frac{\mathrm{d}}{\mathrm{d}z}\tilde{G}=-\frac{\alpha\pi\lambda}{2}g^{\mathrm{s}}(\lambda)\mathrm{e}^{\Psi_+(\lambda,z)}\begin{pmatrix} n_+ & a\sigma_-+\bar{b}n_- \\ \bar{a}\bar{\sigma}_-+bn_- & -n_- \end{pmatrix}\mathrm{e}^{-\Psi_-(\lambda,z)} . \tag{2.25}$$

On the other hand, when G in the form (2.20) is inserted into (2.19*b*), it follows that

$$\tilde{G}(\lambda,z)=\mathrm{e}^{\Psi_+(\lambda,z)}G_0(\lambda,z)\mathrm{e}^{-\Psi_-(\lambda,z)} . \tag{2.26}$$

If $\tilde{G}$ given by (2.26) is used in (2.25), furthermore we find that

$$\frac{\mathrm{d}}{\mathrm{d}z}\Psi_+G_0+\frac{\mathrm{d}}{\mathrm{d}z}G_0-G_0\frac{\mathrm{d}}{\mathrm{d}z}\Psi_-=-\frac{\alpha\pi\lambda}{2}g^{\mathrm{s}}(\lambda)\begin{pmatrix} n_+ & a\sigma_-+\bar{b}n_- \\ \bar{a}\bar{\sigma}_-+bn_- & -n_- \end{pmatrix} . \tag{2.27}$$

Thus far the only assumption we have made is that Ψ is a piecewise analytical function of λ. In the following we shall seek Ψ in the form of a diagonal matrix

$$\Psi(\lambda,z)=\mathrm{diag}\left(\psi_1(\lambda,z),\psi_2(\lambda,z)\right) . \tag{2.28}$$

Equation (2.27) can easily be solved and we find

$$\psi_1(\lambda, z) = -\frac{i\lambda}{2}z + \frac{i\alpha}{4}\int_0^z \left\langle\!\left\langle \omega\frac{n_+}{\omega-\lambda}\right\rangle\!\right\rangle dz'$$
$$= -\frac{i\lambda}{2}z + \frac{i\alpha}{4}\int_0^z \left\langle\!\left\langle \omega\frac{n_-}{\omega-\lambda}\right\rangle\!\right\rangle dz' + \Omega(\lambda, z) \tag{2.29}$$
$$\psi_2(\lambda, z) = \frac{i\lambda}{2}z - \frac{i\alpha}{4}\int_0^z \left\langle\!\left\langle \omega\frac{n_-}{\omega-\lambda}\right\rangle\!\right\rangle dz' .$$

The result (2.29) can now be used to find the general solution of the RMB system with the chosen initial conditions.

If we differentiate (2.22) with respect to time ($\tilde{G}$ does not depend on time), we find that ($\operatorname{Im}\lambda = 0$)

$$\frac{\partial F_-}{\partial t}F_-^{-1} = \frac{\partial F_+}{\partial t}F_+^{-1} . \tag{2.30}$$

On the other hand, the asymptotic behaviour in the limit $|\lambda| \to \infty$ of matrices $F_\pm$ can be shown to be

$$F_\pm = \left(I + \frac{F_1^\pm}{\lambda} + O\left(\frac{1}{|\lambda|^2}\right)\right)\exp\left\{-\frac{i\lambda}{2}(t-z)\sigma_3\right\} \tag{2.31}$$

where I is the identity matrix and $F_1^\pm$ are matrix coefficients.

Application of Liouville's theorem to (2.30) together with (2.31) gives the result

$$U = -\tfrac{1}{2}i\lambda\sigma_3 + \tfrac{1}{2}i[\sigma_3, F_1^\pm] . \tag{2.32}$$

This result also means that

$$\sigma_1 E(t, z) = [\sigma_3, F_1^\pm] . \tag{2.33}$$

By differentiating (2.22) with respect to z, we find that

$$\frac{\partial F_-}{\partial z}F_-^{-1} = \frac{\partial F_+}{\partial z}F_+^{-1} + F_+\frac{d}{dz}\tilde{G}(\lambda, z)F_-^{-1} . \tag{2.34}$$

The last term in this equation can be considered as a jump on the real axis of a piecewise analytical function $\frac{\partial F}{\partial z}F^{-1}$. It follows that this function must have the form

$$\frac{\partial F}{\partial z}F^{-1} = \frac{i\lambda}{2}\sigma_3 - \frac{iE}{2}\sigma_1 - \frac{1}{2\pi i}\int_{-\infty}^{\infty} d\omega\, \frac{F_+(\omega)\frac{d}{dz}\tilde{G}(\omega)F_-^{-1}(\omega)}{\omega-\lambda} . \tag{2.35}$$

Comparison with (2.2*b*) of this result leads now to an expression for $\hat{\rho}(\omega, t, z)$, namely

$$\hat{\rho}(\omega, t, z) = -\frac{2}{\alpha\pi\omega g^s(\omega)}F_+\frac{d}{dz}\tilde{G}F_-^{-1} \tag{2.36}$$

and using the definitions (2.21) and the result (2.25) we find further that

$$\hat{\rho} = G_+^{-1}\begin{pmatrix} n_+ & a\sigma_- + \bar{b}n_- \\ \bar{a}\bar{\sigma}_- + bn_- & -n_- \end{pmatrix}G_-^{-1} . \tag{2.37}$$

The two results, (2.33) and (2.37), now provide the general solution of the RMB system.

Before going into the Darboux transformation for the RMB system, we shall discuss briefly the trace formulae which will be needed later.

The trace formulae generated by the spectral problem (2.1) with (2.2) take the form

$$I_k = c_k \qquad k = 1, 2, \ldots \tag{2.38}$$

where the I_k's are the Zakharov–Shabat functionals [4]. The first three of them are

$$I_1 = \frac{1}{4}\int_{-\infty}^{\infty} \mathrm{d}t\, E^2 \qquad I_2 = 0 \qquad I_3 = -\frac{1}{4}\int_{-\infty}^{\infty} \mathrm{d}t\,(EE_{tt} + \tfrac{1}{4}E^4). \tag{2.39}$$

The constants c_k can be found by expanding $\ln a(\lambda, z)$ for $|\lambda| \to \infty$ in terms of λ^{-1}

$$\begin{aligned} c_{2m} &= 0 \\ c_{2m-1} &= -\frac{2\mathrm{i}}{2m-1}\left\{\sum_{j=1}^{n_1}(\mathrm{i}\kappa_j)^{2m-1} - \sum_{j=n_1+1}^{n_1+n_2}(\bar{\lambda}_j^{2m-1} - \lambda_j^{2m-1})\right\} \\ &\quad + \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\omega\, \omega^{2m-1}\ln\left(1 + |r(\omega, z)|^2\right) \qquad m = 1, 2, \ldots. \end{aligned} \tag{2.40}$$

From (2.39) and (2.40) the evolution equations of the Zakharov–Shabat functionals can be deduced straightforwardly

$$\frac{\mathrm{d}}{\mathrm{d}z} I_{2m-1} = \frac{\alpha}{4}\left\langle\!\!\left\langle \frac{\omega^{2m-1}}{1 + |r|^2}(2|r|^2 n_- + r\bar{\sigma}_- + \bar{r}\sigma_-)\right\rangle\!\!\right\rangle. \tag{2.41}$$

Finally it is worth pointing out that the RMB system is one of those systems [5] which cannot have other constants of motion than its spectrum.

3. Darboux transformation for the RMB system

In the derivation of the one-soliton (or a many-soliton) solution of the RMB system on an arbitrary background we shall use the Darboux transformation method which has proved to be [6–8] a very elegant and powerful method. In this section we shall therefore formulate the Darboux transformation for the RMB system. To this end it is convenient to reformulate the auxiliary linear problem (2.1) with (2.2) such that it is expressed in the following way:

$$\Phi_\xi = U_1 \Phi\, \Lambda + U_0 \Phi \tag{3.1}$$

$$\Phi_\tau = \langle M_1 \Phi P_1 + M_2 \Phi P_2 \rangle \tag{3.2}$$

where $\xi = t - z$, $\tau = \alpha z$. We have also introduced above a diagonal spectral parameter matrix

$$\Lambda = \mathrm{diag}(\lambda_1, \lambda_2) \tag{3.2a}$$

and the corresponding matrix of eigenfunctions

$$\Phi = \begin{pmatrix} \psi_1(\lambda_1) & \psi_2(\lambda_2) \\ \phi_1(\lambda_1) & \phi_2(\lambda_2) \end{pmatrix} \tag{3.2b}$$

where $(\psi_i(\lambda_i), \phi_i(\lambda_i))^{\mathrm{T}}$ are solutions of the linear system (2.1) with (2.2).

We have likewise defined the matrices

$$U_0 = \mathrm{i}E\sigma_1 \qquad U_1 = -\mathrm{i}\sigma_3 \tag{3.3a}$$

$$M_{1,2} = \frac{\mathrm{i}\omega}{8}\begin{pmatrix} n & \pm u - \mathrm{i}w \\ \pm u + \mathrm{i}w & -n \end{pmatrix} \tag{3.3b}$$

$$P_{1,2} = \mathrm{diag}\left((\lambda_1 \mp \omega/2)^{-1},\ (\lambda_2 \mp \omega/2)^{-1}\right). \tag{3.3c}$$

The idea of the Darboux transformation method is to construct a linear transformation (i.e. the Darboux transformation) for which the auxiliary linear problem (3.1) is covariant and to thereby determine the so-called dressing formulae. In this case we seek the Darboux transformation in the form

$$\Phi\,[N] = \sum_{j=0}^{N} S_j \Phi\, \Lambda^{N-j} \qquad S_0 = I\,. \tag{3.4}$$

After straightforward but tedious algebra, which employs use of properties of the integrals of Cauchy type, we find

$$E[N] = E - 2v^{(2)}/\Delta \tag{3.5a}$$

$$M_{1,2}[N] = \varrho_{1,2} M_{1,2} \varrho_{1,2}^{-1} \tag{3.5b}$$

where E is a background electric field introduced in (1.2) and

$$\begin{aligned} \varrho_1 &= \sum_{j=0}^{N} (\omega/2)^{N-j} S_j \\ \varrho_2(\omega) &= \varrho_1(-\omega) \\ v^{(k)} &= \det(v_{nm}^{(k)}) \qquad \Delta = \det(\Delta_{nm})\,. \end{aligned} \tag{3.6}$$

Here $\left(v_{nm}^{(k)}\right)$ and $\left(\Delta_{nm}\right)$ are matrices to be defined below.

The transformations (3.5) describe N-soliton solutions for any choice of the 'background' variables E, n, u and v. Equation (3.5a) has also been reported in [8], where it was derived as a general result valid for any integrable system of the AKNS class.

Note that the determinant of the transformation matrix, $\det(\sum_{j=0}^{N} S_j \Phi\, \Lambda^{N-j})$, of the Darboux transformation (3.4) has $2N$ zeros λ_k ($k = 1, 2, \ldots, 2N$), and that there are $2N$ related eigenfunctions with components $\psi_k = \psi(\lambda_k)$ and $\phi_k = \phi(\lambda_k)$. These eigenfunctions form vector solutions $(\psi_k, \phi_k)^{\mathrm{T}}$ of the auxiliary linear problem (3.1). Not all of these solutions can be independent, however, because the involution conditions (2.6) must be satisfied. We find in particular that imposing (2.6) leads to conditions

$$\begin{aligned} &\psi_{2k} = -\bar{\phi}_{2k-1} \qquad \phi_{2k} = \bar{\psi}_{2k-1} \qquad \lambda_{2k} = \bar{\lambda}_{2k-1} \\ &\psi_{2k-1} = \phi_{2k} \qquad \phi_{2k-1} = \psi_{2k} \qquad \lambda_{2k-1} = -\lambda_{2k}\,. \end{aligned} \tag{3.7}$$

These conditions mean that we must choose eigenvalues and eigenfunctions which satisfy, e.g.

$$\psi_{2k-1} = \bar{\psi}_{2k-1} \qquad \phi_{2k-1} = -\bar{\phi}_{2k-1} \qquad \lambda_{2k-1} = -\bar{\lambda}_{2k-1}\,. \tag{3.8}$$

The solution (3.5) of the RMB system can be expressed in terms of $2N \times 2N$ matrices (Δ_{mn}), $\left(v_{nm}^{(k)}\right)$ and $\left(\delta_{nm}^{(ik)}\right)$ $\quad n, m = 1, 2, \ldots, 2N$, whose elements are functions of the above eigenvalues and eigenfunctions. We find that

$$\Delta_{nm} = \begin{cases} \lambda_n^{N-k}\psi_n & m = 2k-1 \\ \lambda_n^{N-k}\phi_n & m = 2k \end{cases} \qquad k = 1, 2, \ldots, N \tag{3.9a}$$

$$v_{nm}^{(k)} = \begin{cases} \Delta_{nm} & m \neq k \\ \lambda_n^N \psi_n & m = k \end{cases} \qquad k = 1, 2, \ldots, 2N. \tag{3.9b}$$

The elements of the four matrices $\left(\delta_{nm}^{(ik)}\right)$ $\quad i, k = 1, 2$, are found to be

$$\delta_{nm}^{(11)} = \begin{cases} \left(\lambda_n^N - (\omega/2)^N\right)\psi_n & m = 1 \\ \lambda_n^{N-k}\phi_n & m = 2k \\ \lambda_n^{N-j}\left((\omega/2)^{j-1} - \lambda_n^{j-1}\right)\psi_n & m = 2j-1 \end{cases} \tag{3.9c}$$

$$\delta_{nm}^{(12)} = \begin{cases} \lambda_n^{N-k}\psi_n & m = 2k-1 \\ \lambda_n^N \psi_n & m = 2 \\ \lambda_n^{N-j}\left((\omega/2)^{j-1} - \lambda_n^{j-1}\right)\phi_n & m = 2j \end{cases} \tag{3.9d}$$

$$\delta_{nm}^{(21)} = \begin{cases} \lambda_n^N \phi_n & m = 1 \\ \lambda_n^{N-k}\phi_n & m = 2k \\ \lambda_n^{N-j}\left((\omega/2)^{j-1} - \lambda_n^{j-1}\right)\psi_n & m = 2j-1 \end{cases} \tag{3.9e}$$

$$\delta_{nm}^{(22)} = \begin{cases} \lambda_n^{N-k}\psi_n & m = 2k-1 \\ \left(\lambda_n^N - (\omega/2)^N\right)\phi_n & m = 2 \\ \lambda_n^{N-j}\left((\omega/2)^{j-1} - \lambda_n^{j-1}\right)\phi_n & m = 2j \end{cases} \tag{3.9f}$$

with $k = 1, 2, \ldots, N$ and $j = 1, 2, \ldots, N$.

The solution for the matrix $\varrho_1(\omega)$ can now be written in the form

$$\varrho_1 = -\frac{1}{\Delta}\left(\frac{\omega}{2}\right)^{-(N-1)(N-2)/2} \begin{pmatrix} \delta^{(11)} & \delta^{(12)} \\ \delta^{(21)} & \delta^{(22)} \end{pmatrix} \tag{3.10}$$

where $\delta^{(ik)} = \det(\delta_{nm}^{(ik)})$. With this the solution (3.5) is completely specified. In practice the way to proceed is to find a solution of the original RMB system (e.g. the trivial solution with $E = 0$) and then find new solutions through iterative dressings by transformation (3.5). For completeness we quote here the detailed form of the one-step Darboux transformations, i.e. (3.5) for $N = 1$

$$\begin{aligned} E[1] &= E + 4\frac{D}{\Delta} \\ n[1] &= \frac{4}{(4\lambda^2 - \omega^2)\Delta^2}[(H^2 + D^2 - \tfrac{1}{4}\omega^2\Delta^2)n - 2iHDv - \omega\Delta Du] \\ v[1] &= \frac{4}{(4\lambda^2 - \omega^2)\Delta^2}[-2iHDn - (H^2 + D^2 + \tfrac{1}{4}\omega^2\Delta^2)v + i\omega\Delta Hu] \\ u[1] &= \frac{4}{(4\lambda^2 - \omega^2)\Delta}[\omega Dn - i\omega Hv - (\lambda^2 + \tfrac{1}{4}\omega^2)u\Delta] \end{aligned} \tag{3.11}$$

where

$$H = \lambda(\psi^2 + \phi^2) \qquad D = 2\lambda\psi\phi \qquad \Delta = \psi^2 - \phi^2$$
$$\psi = \bar{\psi} \qquad \phi = -\bar{\phi} \qquad \lambda = -\bar{\lambda}.$$

4. Interaction of a pulse of light with an arbitrary background

In this section we shall apply the results of the previous sections to construction of the one-soliton solution of the RMB system on an arbitrary background. It is worth noting that in this section we are considering a background electric field which is rapidly decreasing for $|t| \to \infty$. This means that for the construction of the soliton solution we could also use the Riemann–Hilbert formalism developed in section 2 for $E \in L_1(-\infty, \infty)$. But due to its pure algebraic nature, the Darboux transformation method does not require application of the full machinery of the inverse scattering method nor does it impose strict limitations on the functional classes of considered solutions. Thus the domain of applicability of the formulae of section 3 is rather broad and includes for example rational and periodic cases [9]. Hence our approach can easily be applied to background fields relevant in a variety of physical problems. Let us solve first the auxiliary linear problem (2.1) with (2.2) for arbitrary 'background' variables $E(t, z)$, $n(t, z, \omega)$, $u(t, z, \omega)$ and $v(t, z, \omega)$.

The solution Φ of (2.1) will be sought in the form

$$\Phi = (S_+/a)\exp(-\Psi)C \tag{4.1}$$

where $\Psi(\lambda, z)$ is an integral of Cauchy type and is of the form of (2.28), S_+ is defined by (2.19*d*) and C is a constant matrix.

The Jost functions $T_\pm(\lambda, t, z)$ which are related with the background fields can be found in the form of asymptotic expansions in terms of λ^{-1}. It is convenient to express these asymptotic expansions as

$$T_\pm(\lambda, t) = (I + \mathcal{W}(t, \lambda))\exp\left(-\frac{i\lambda}{2}\sigma_3 t + \frac{i}{2}\int_{\pm\infty}^{t} E\sigma_1\mathcal{W}\,dt'\right) \tag{4.2}$$

where $\mathcal{W}(t, \lambda)$ is an antidiagonal matrix which has for $|\lambda| \to \infty$ an asymptotic expansion

$$\mathcal{W}(t, \lambda) = \sum_{n=1}^{\infty}\frac{\mathcal{W}_n(t)}{\lambda^n} + O(|\lambda|^{-\infty}). \tag{4.3}$$

It is plain that in order to comply with involutions (2.6), $\mathcal{W}(t, \lambda)$ must be of the form

$$\begin{aligned} \mathcal{W}(t, \lambda) &= \bar{w}(t, \bar{\lambda})\sigma_+ - w(t, \lambda)\sigma_- \\ w(t, \lambda) &= -\bar{w}(t, -\bar{\lambda}) \end{aligned} \tag{4.4}$$

where

$$\sigma_\pm = \tfrac{1}{2}(\sigma_1 \pm i\sigma_2)$$

and $w(t,\lambda)$ is a function which also has an asymptotic expansion which can be found from (4.3). In order to find the one-soliton solution we apply next the one step Darboux transformation formula given in (3.11). For a soliton solution one must have the spectral parameter on the positive imaginary λ axis, $\lambda = \mathrm{i}\kappa,\ \kappa > 0$. Let us suppose for simplicity that $i\kappa$ does not belong to the discrete spectrum of the 'background' field $E(t,z)$ which means that $a(\mathrm{i}\kappa, 0) \neq 0$. The first column $\Phi^{(1)} = (\psi,\phi)^{\mathrm{T}}$ of the matrix function Φ appearing in (2.1) and (4.1) now takes the form

$$\begin{aligned}\psi &= \frac{1}{a(\mathrm{i}\kappa,0)}\left(c_1 \exp\chi_1 + c_2\bar{w}(-\mathrm{i}\kappa,t)\exp\chi_2\right)\\ \phi &= \frac{1}{a(\mathrm{i}\kappa,0)}\left(-c_1 w(\mathrm{i}\kappa,t)\exp\chi_1 + c_2\exp\chi_2\right)\end{aligned} \tag{4.5}$$

where

$$\begin{aligned}\chi_1 &= \frac{\theta}{2} - \frac{\mathrm{i}}{2}\int_{-\infty}^{t} E(t')w(\mathrm{i}\kappa,t')\,\mathrm{d}t'\\ \chi_2 &= -\frac{\theta}{2} - \frac{\mathrm{i}}{2}\int_{t}^{\infty} E(t')\bar{w}(-\mathrm{i}\kappa,t')\,\mathrm{d}t' + \Omega(\mathrm{i}\kappa,z)\\ \theta &= \kappa\left(t - t_0 - z + \alpha\int_0^z \left\langle\frac{\omega n_-(\omega,z')}{\omega^2+\kappa^2}\right\rangle \mathrm{d}z'\right)\\ \Omega(\mathrm{i}\kappa,z) &= \frac{\kappa}{\pi}\int_0^\infty \mathrm{d}\omega\,(\omega^2+\kappa^2)^{-1}\ln\frac{1+|r(\omega,z)|^2}{1+|r(\omega,0)|^2}\,.\end{aligned} \tag{4.6}$$

Note that in (4.6) θ is the phase of an 'undisturbed' soliton in the absence of any 'background' field. If (4.5) is now used as the starting solution in the Darboux transformation (3.11), the one-soliton solution of the RMB system on the arbitrary background E is found to be

$$E[1] = E + 4\mathrm{i}\kappa\frac{\psi\phi}{\psi^2-\phi^2} \tag{4.7}$$

or in a more elaborated form,

$$E[1] = E - \frac{2\kappa}{\cosh\chi}\left(1 + \frac{Q_1}{\kappa} + \frac{Q_2}{\kappa^2} + \mathcal{O}\left(\frac{1}{\kappa^3}\right)\right) \tag{4.8}$$

where

$$Q_1 = E\frac{\sinh^2\chi}{\cosh\chi} \qquad Q_2 = \frac{E^2}{4}\left(-5 + \frac{4}{\cosh^2\chi}\right) - E_t\sinh\chi\,.$$

The phase χ is given by

$$\begin{aligned}\chi &= \theta - \frac{\mathrm{i}}{2}\int_{-\infty}^{t} E(t')w(\mathrm{i}\kappa,t')\,\mathrm{d}t' + \frac{\mathrm{i}}{2}\int_{t}^{\infty} E(t')\bar{w}(-\mathrm{i}\kappa,t')\,\mathrm{d}t' - \Omega(\mathrm{i}\kappa,z)\\ &= \kappa(t-t_0-z) + \frac{1}{4\kappa}\int_{-\infty}^{\infty}\mathrm{sgn}(t'-t)E^2(t')\,\mathrm{d}t'\\ &\quad - \frac{1}{\kappa\pi}\int_0^\infty d\omega\ln\frac{1+|r(\omega,z)|^2}{1+|r(\omega,0)|^2}\\ &\quad + \frac{\alpha}{\kappa}\int_0^z\langle\omega n_-\rangle\,\mathrm{d}z' + \mathcal{O}\left(\frac{1}{\kappa^2}\right).\end{aligned} \tag{4.9}$$

The second term on the right-hand side of expression (4.8) can be interpreted as a soliton perturbed by the arbitrary background. Other terms in this expansion describe the soliton-background interaction. The phase velocity of the pulse is given by the condition $\mathrm{d}\chi = 0$ and reads

$$\frac{\mathrm{d}z}{\mathrm{d}t} = 1 + \frac{\alpha}{\kappa^2}\langle \omega n_- \rangle - \frac{1}{2\kappa^2}\frac{\partial}{\partial z}\int_{-\infty}^{t} E^2\,\mathrm{d}t' - \frac{1}{2\kappa^2}E^2 + \mathcal{O}\left(\frac{1}{\kappa^3}\right). \tag{4.10}$$

Here the first two terms on the right side are the same as in the asymptotic expansion for the phase velocity of the free soliton, and the rest of the terms describe the influence of the background at the given spacetime point. In order to get an idea of how the solution looks like in the asymptotically far region, we shall work out its asymptotic behaviour at $t, z \to \infty$. We shall only consider the case of attenuation, i.e. $n_-(\omega, z) < 0$.

The limit $t, z \to \infty$ is taken in such a way that the phase θ of the undisturbed soliton remains constant. In this way we find

$$\begin{aligned} \psi &\sim \frac{c_1}{a(\mathrm{i}\kappa, 0)} \exp(\theta/2 + I(\mathrm{i}\kappa, \infty)) \\ \phi &\sim \frac{c_2}{a(\mathrm{i}\kappa, 0)} \exp(-\theta/2 + \Omega(\mathrm{i}\kappa, \infty)) \end{aligned} \tag{4.11}$$

where

$$I(\mathrm{i}\kappa, \infty) = \sum_{m=1}^{\infty} (-1)^m \frac{I_{2m-1}(\infty)}{\kappa^{2m-1}} \tag{4.12}$$

and $I_{2m-1}(\infty)$ are limiting values of Zakharov–Shabat functionals for $z \to \infty$. The asymptotic form of the soliton solution (4.7) is now

$$E[1] \sim -2\kappa\left\{\cosh\left(\theta - \ln a(\mathrm{i}\kappa, 0) - \frac{\kappa}{\pi}\int_0^{\infty} \mathrm{d}\omega\, \frac{\ln(1 + |r(\omega, \infty)|^2)}{\omega^2 + \kappa^2}\right)\right\}^{-1}. \tag{4.13}$$

This is the form of the soliton when it has traversed the whole space filled by the arbitrary background. From the analytical expression (4.13) we can then easily deduce the phase shift $\Delta\Phi$ experienced by the soliton due to the background

$$\Delta\Phi = -\ln a(\mathrm{i}\kappa, 0) - \frac{\kappa}{\pi}\int_0^{\infty} \mathrm{d}\omega\, \frac{\ln(1 + |r(\omega, \infty)|^2)}{\omega^2 + \kappa^2}. \tag{4.14}$$

The first term on the right-hand side of (4.14) is the contribution of the usual 'causal' part of the radiation field (due to the initial condition $E_0(t)$). The second term is the contribution of the 'spontaneous' radiation field which has not been reported before. The asymptotic value of the reflection coefficient $r(\omega, \infty)$ can be expressed in terms of the boundary conditions. For simplicity we quote here its value in the case when the boundary values of polarization and inversion are independent of z, i.e. $\sigma_- = \sigma_-(\omega)$, $n_- = n_-(\omega) < 0$:

$$\begin{aligned} |r(\omega, \infty)|^2 = \left|\frac{\alpha\pi\omega}{2} g^s(\omega)\sigma_-(\omega)\right|^2 &\left\{\left|\frac{\alpha\pi\omega}{2} g^s(\omega) n_-(\omega)\right|^2\right. \\ &\left. + \left[\omega - \frac{\alpha}{2}\mathcal{P}\int_{-\infty}^{\infty} \mathrm{d}\omega'\, g^s(\omega')\frac{\omega' n_-(\omega')}{\omega' - \omega}\right]^2\right\}^{-1}. \end{aligned} \tag{4.15}$$

Many-soliton and breather solutions of the RMB system on an arbitrary background can be derived in essentially the same way as the one-soliton solution reported here, but the algebraic manipulations become rather tedious. Also in many experimental situations the density of solitons is so low that the independent-soliton approximation where soliton–soliton interactions can be neglected provides a valid description of the problem, and the one-soliton solution (4.7) is applicable.

References

[1] Bullough R K, Jack P M, Kitchenside P W and Saunders R 1979 *Phys. Scr.* **20** 364
[2] Ablowitz M J, Kaup D J, Newell A C and Segur H 1974 *J. Math. Phys.* **15** 1852
[3] Gabitov I R, Zakharov V E and Mikhailov A V 1985 *Theor. Math. Phys.* **63** 11
[4] Faddeev L D and Takhtajan L A 1987 1988 *Hamiltonian Methods in the Theory of Solitons* (Berlin: Springer)
[5] Newell A C 1980 *Solitons (Topics in Current Physics)* ed R K Bullough and P J Caudrey (Berlin: Springer)
[6] Matveev V B, Salle M A and Rybin A V 1988 *Inverse Problems* **4** 173
[7] Rybin A V 1988 *Problems in Quantum Field Theory and Statistical Physics* vol 8, *Zap. Nauchn. Semin. LOMI* **169** 141 (in Russian)
[8] Neugebauer G and Meinel R 1984 *Phys. Lett.* **100A** 467
[9] Its A R, Rybin A V and Salle M A 1988 *Theor. Math. Phys.* **74** 20–32